Chern-Simons Gravity: From 2+1 to 2n+1Dimensions [*]

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These lectures provide an elementary introduction to Chern Simons Gravity and Supergravity in $d = 2n + 1$ dimensions.

I. INTRODUCTION

The present situation of High Energy Physics is both exciting and paradoxical. On one side we have a monumental theoretical machinery built on the most beautiful ideas –strings, membranes, dualities, $M-$theory, AdS-CFT correspondence–, with hardly any experimentally testable predictions for the near future. On the other hand, there is a growing corpus of puzzling observations coming mostly from astrophysics, including gamma ray bursts, missing mass, microwave inhomogeneities, indications of an accelerated expansion, which defy present standard wisdom about the universe, acquired through observations of our more immediate neighborhood. The new observations are for the most part surprising, and one could expect by now that the surprises will continue. In an effort to cope with the new data, a number of exotic proposals have been put forward. Thus, in recent times notions such as *quintessence* and other nonminimal extensions of General Relativity (**GR**) including a nonvanishing cosmological constant, and extra dimensions at macroscopic scales have gained acceptance in respectable journals.

In this situation, a reasonable attitude for a theoretical physicist would be to critically examine all consistent alternatives to our time-honored foundations. In this vein, here we discuss a class of gravity theories which share the essential geometric foundation of GR:

- **a**) General covariance

- **b**) Second order field equations for the metric

  At the same time, other features of GR are relaxed:

- **c**) Spacetime is allowed to have any number of dimensions

- **d**) Spacetime is not necessarily asymptotically flat

- **e**) The action is extremized under independent variations of the metric and affine connection

Condition (c) is absolutely necessary in the context of string theory and (d) is most natural in the light of the recently discovered AdS-CFT correspondence.

Requirement (e), on the other hand, allows for independent propagation of the metric and affine structures of spacetime. This is satisfactory in view of the fact that the metric and affine features of spacetime are geometrically independent: one has to do with the measurement of distances while the other relates to parallel transport. A natural way to implement condition (c) is to use the first order formalism, where the independent fields are the vielbein one-form $e^a = e^a_\mu dx^\mu$, and the spin connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$.

The family of gravity theories obtained with these postulates where studied by Lovelock in the early 70's [1]. The five dimensional case had already been discussed by Lanczos in the late 30's [2]. More recently, Zwiebach [3] and Zumino [4] showed the Lanczos-Lovelock (**LL**) theories to be appropriate to describe the effective low-energy gravity theory found in the weak coupling limit of string theory.

In their simplest version, the **LL** theories have the same fields, symmetries and local degrees of freedom as ordinary gravity. The action is a polynomial of degree $[d/2]$ in curvature[1], which can also be written in terms of the Riemann curvature $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$ and the vielbein as[2]

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[1]Here $[x]$ is the integer part of x.
[2]Wedge product between forms is understood throughout.

$$I_G = \kappa \int \sum_{p=0}^{[d/2]} \alpha_p L^{(p)}, \tag{1}$$

where α_p are arbitrary constants, and $L^{(p)}$ is given by

$$L^{(p)} = \epsilon_{a_1 \cdots a_d} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d}. \tag{2}$$

What makes the LL theories so special is the fact that they comply with the requirement b) above. An arbitrary Lagrangian density constructed with the metric and curvature tensors, on the contrary, would give rise to fourth order field equations for the metric, in general. A powerful reason to choose the Einstein-Hilbert (**EH**)action in four dimensions is that the Einstein equations are second order. This feature of the EH action stems from the fact that the EH Lagrangian in *two dimensions* is the density of a topological invariant: the Euler characteristic of the manifold. Similarly, the LL theories in d-dimensions are linear combinations (with arbitrary coefficients) of the Euler densities of all dimensions below d. Thus, General Relativity is a particular case of LL theory.

A. Drawbacks of the Generic LL Action

In spite of their nice features, the LL theories suffer from an original sin: they are endowed with a collection of indeterminate dimensionful parameters α_p, $p = 1, ..., \left[\frac{d}{2}\right]$. This has two puzzling consequences already at the classical level:

- (**i**) The theories have a large number of physical parameters which should be experimentally determined. This would make gravity less interesting as a fundamental theory, because it would have more natural constants, like G_{Newton} and the cosmological constant Λ, to be adjusted.

- (**ii**) The field equations admit solutions which have indeterminate spacelike dependence and timelike evolution. This results from the fact that the field equations are polynomials of degree p in the derivatives of the metric (∂g). Thus, the values of the velocity can jump between different roots of the equation arbitrarily and still extremize the action [5], [6].

A consequence of (ii) is the fact that the Legendre transformation from the Lagrangian to the Hamiltonian cannot be performed in general, making the canonical quantization program ill-defined. Another problem related with the second issue is the existence of several vacua with different topologies. This could be an interesting novel feature of these theories, were it not for the fact that the perturbation expansions around the different vacua give rise to completely different theories and typically contain ghosts [9].

There is one more reason to find the presence of the large number of coefficients in the Lagrangian undesirable. The bare α_p's are dimensionful constants and therefore could receive quantum corrections beyond control, making the possibility of constructing a quantum theory of gravity even more remote than in standard GR. This would be so unless the values of these constants are protected by some fundamental symmetry, like the zero mass for the photon, or the equal number of quarks and leptons, which are "protected" by gauge invariance.

Thus, it would be interesting to find a "natural" way to fix the α's. Moreover, if the criterion that fixes these coefficients is based on some symmetry principle, and possibly protect them from renormalization with a reasonable symmetry principle. In this scenario, one finds two special families, which stand out among all LL theories. They correspond to a Born-Infeld -like theory in even dimensions, and the Chern-Simons theory for the anti-de Sitter (**AdS**) gauge group for odd D. In section **III** we review these theories in general. In section **IV** the CS case and their supersymmetric extensions are analyzed in greater depth. The next section is just a cursory review of nonabelian CS theories which can be skipped by the experts and by those eager to get to the juicy stuff.

II. CHERN-SIMONS THEORY IN 3 DIMENSIONS

Chern-Simons ($\mathbf{CS}^3$) theory has a curious history. It was discovered in the context of anomalies in the 70's and used as a rather exotic toy model for gauge systems in 2+1 dimensions ever since [10] . It was only by the mid 80's

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[3]There is a vast literature on this subject. For an extensive treatment of CS theories, see e.g., [7], or [8].

that it was realized that ordinary Einstein gravity in 2+1 dimensions is a natural example of a **CS** system, especially through the work of Witten [11]. As it turns out, **CS** systems are more conspicuous than it might seem at first sight. General Relativity in 2+1 dimensions (with or without cosmological constant) is a **CS** system (for ISO(2,1) or SO(2,2) groups, respectively); any ordinary mechanical system in Hamiltonian form can be viewed as an abelian **CS** system in 0+1 dimensions [12]. This way of looking at mechanical systems is not completely absurd and it even sheds some light into ancient problems such as the justification for the old quantization rule of Bohr and Sommerfeld.

In retrospect, we can see that the key to the construction of the **CS** form (in three dimensions) is the following: the Pontryagin form

$$P = \text{Tr}[\mathbf{F} \wedge \mathbf{F}], \tag{3}$$

is **closed**

$$dP = 0. \tag{4}$$

By Poincaré's lemma, P is **locally exact**, that is, it is always possible to write it in an open neighborhood as the exterior derivative of a 3-form

$$P = dL. \tag{5}$$

Thus, the 3-form L is the **Chern-Simons** Lagrangian. Clearly, this idea can be generalized to higher (odd) dimensions, and for other integral topological invariants, like the Euler characteristic. This is precisely where the connection with the LL theory can be found: if one asks, *what is the 3-form whose exterior derivative is the 4-dimensional Euler density?*, the answer is the Einstein-Hilbert action with nonzero cosmological constant.We now briefly review a few

facts about standard 3-dimensional **CS** systems.

The idea is to find a three-form L_{CS} such that

$$dL_{CS} = \text{Tr}[\mathbf{F} \wedge \mathbf{F}], \tag{6}$$

where

$$\mathbf{F} = \mathbf{dA} + \mathbf{A} \wedge \mathbf{A} \tag{7}$$

is the curvature (field strength) in the adjoint representation and $\mathbf{A}$ is a Lie algebra-valued connection1-form.

Let $\mathbf{G}$ be the gauge group and G its Lie algebra generated by the matrices $\mathbf{T}_a$, such that $[\mathbf{T}_a, \mathbf{T}_b] = C_{ab}^c \mathbf{T}_c$. Under the action of the gauge group, the connection

$$\mathbf{A} = A_\mu^a \mathbf{T}_a dx^\mu$$

transforms as

$$\mathbf{A} \rightarrow \mathbf{A}' = g^{-1}\mathbf{A}g + g^{-1}dg, \tag{8}$$

where the 0-form $g(x)$ is an element of G. Then, the curvature changes as

$$\mathbf{F} \rightarrow \mathbf{F}' = g^{-1}\mathbf{F}g, \tag{9}$$

and it is easily shown, using the cyclic property of the trace, that $\text{Tr}[\mathbf{F} \wedge \mathbf{F}]$ is invariant under (8, 9). From (6), the **CS** Lagrangian is found to be

$$L_{CS} = \text{Tr}[\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}]. \tag{10}$$

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A. Gauge Invariance

It is easily checked that the **CS** action is invariant under gauge transformations. First observe that under a gauge transformation of the form (8), the right hand side of (6) does not change and therefore $\delta(dL_{CS}) = 0$

$$d\delta L_{CS} = 0.$$

In other words, under a variation of the fields that leaves the Pontryagin form invariant, the **CS** Lagrangian changes by a closed form. Thus, provided the change δL_{CS} approaches zero sufficiently fast at the spacetime boundary, the **CS** action should be invariant as well. Substituting (8) in (10) one finds

$$L_{CS}(\mathbf{A}') = L_{CS}(\mathbf{A}) - d\text{Tr}[dgg^{-1}\mathbf{A}] - \frac{1}{3}\text{Tr}[(g^{-1}dg)^3]. \tag{11}$$

Then the action changes as

$$I_{CS}[\mathbf{A}'] = I_{CS}[\mathbf{A}] - \int_{\partial M} \text{Tr}[dgg^{-1}\mathbf{A}] - \frac{1}{3}\int_{M} \text{Tr}[(gdg^{-1})^3]. \tag{12}$$

This raises an important issue: is the action *really* invariant under the gauge transformation (8)? The answer seems to be no, unless $g \to 1$ sufficiently fast to cancel the second term, and some other miracle makes the third term also vanish. The first condition one can always demand because it is part of the rules of the game in any variational problem that the fields satisfy appropriate boundary conditions, and that necessarily restricts the type of field transformations allowed at the spacetime boundary.

The last term in (11) is closed and therefore, provided there are no topological contrivances, this term can be expressed locally as the exterior derivative of some 2-form which depends on $g(x)$. It is obvious that this term cannot be made to vanish simply by imposing some asymptotic condition on the gauge transformation g. In fact, there are some interesting simple examples in which the integral of this last term doesn't vanish. For instance, if the manifold M is topologically a 3-sphere and the gauge group is $SU(2)$, the last term in (12) is $4\pi^2 N$, where N is the **winding number** of the mapping $g : M \to SU(2)$.

The transformation law (12) tells us that the action changes by a surface term and possibly by a functional of $g(x)$. Although none of these terms can alter the field equations, they can change the global properties of the theory, like the definition of conserved charges. They also provide different weight for topologically different configurations in the quantum theory as defined by the path integral. At any rate, the action $I_{CS}[\mathbf{A}]$ is genuinely gauge invariant if the manifold has no boundary ($\partial M = 0$), or the gauge transformation goes to zero at the boundary fast enough, and the topology of the mapping $g : M \to \mathbf{G}$ is trivial. These conditions are met by gauge transformations which are everywhere infinitesimally close to the identity, e.g., $g = 1 + \lambda^a(x)\mathbf{T}_a$, with $\lambda^a(x) << 1$. This is sufficient for most practical purposes in the study of **CS** systems as field theories.

B. Field equations

Now that we have a good action principle for the connection $\mathbf{A}$, it is natural to ask, what does it describe? One way to answer this is to study the field equations. Varying the action yields

$$\delta I_{CS}[\mathbf{A}] = 2\int_{M} \text{Tr}[\mathbf{T}_a\mathbf{T}_b]F^a \wedge \delta A^b - \int_{\partial M} \text{Tr}[\mathbf{T}_a\mathbf{T}_b]A^a \wedge \delta A^b.$$

Here we see that the condition of having an extreme under arbitrary variations δA^b implies

$$F^a = 0, \tag{13}$$

provided the group algebra is such that

$$\gamma_{ab} \equiv \text{Tr}[\mathbf{T}_a\mathbf{T}_b]$$

is a non-singular matrix, which is always the case for a **semisimple** Lie algebra in the adjoint representation and γ_{ab} is the Killing metric.

Semisimple algebras are those which do not contain invariant abelian subalgebras (roughly, those that *cannot* be written as $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{U}$, where $\mathcal{U}$ is abelian). Semisimple algebras correspond to the classical groups $SO(n)$, $SU(n)$,

$Sp(2n)$, and other more exotic choices such as $OSp(n,m)$, $USp(p,q)$, E_8, etc. There is an important exceptional Lie algebra which **is not** semisimple and yet there is a faithful representation for which γ_{ab} is nondegenerate. This is the case of the Poincaré group in 2+1 dimensions, $ISO(2,1)$, whose algebra is $so(2,1) \oplus R^{2,1}$, where $R^{2,1}$ is the group of translations in 2+1 dimensions. This exception to the rule allows writing the Einstein-Hilbert Lagrangian as a **CS** 3-form for the Poincaré group, and is the key to the quantizability of gravity in 2+1 dimensions [11].

The field equations (13) look strikingly simple and not much could be expected from them. In fact, there is a well known result according to which if on a homotopically trivial open set B,

$$d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = 0,$$

then $\mathbf{A}$ can be written as a gauge transformation of a trivial connection,

$$\mathbf{A} = g^{-1}dg \text{ everywhere in } B. \tag{14}$$

In other words, by a gauge transformation it is always possible to take $\mathbf{A} \to 0$ on any small patch of M. Thus, the field configurations described by the equation (13) would be trivial unless there are topological obstructions which prevent (14) from being valid globally throughout the entire manifold M even if it remains valid on any small open set B. This is in fact what happens with gravity in 2+1 dimensions, where the field equations are of the form (13) and still one can have nontrivial solutions such as black holes and gravitational collapse. At any rate, what remains true is the fact that a **CS** system in 2+1 dimensions has no local degrees of freedom that could propagate. In higher dimensions, however, **CS** systems possess propagating local degrees of freedom and the situation is similar to that of a standard gauge theory.

C. Generalization to Higher Dimensions

In spite of their interesting mathematical structure, these three-dimensional theories might seem too unrealistic as models for our world. We are now going to see how these ideas are extended to higher dimensions. The essential ingredient in the construction of CS theories in higher dimensions is the existence of a $2n-$ form

$$Q_{2n}(\mathbf{A}) = \gamma_{a_1\ldots a_n} F^{a_1} \wedge F^{a_2} \wedge \cdots \wedge F^{a_n}, \tag{15}$$

which is closed,

$$dQ_{2n} = 0 \; ,$$

and invariant under a gauge transformation $\mathbf{A} \to \mathbf{A}' = g^{-1}\mathbf{A}g + g^{-1}dg$,

$$Q_{2n}(\mathbf{A}') = Q_{2n}(\mathbf{A}).$$

It is straightforward to show that the invariants of the form

$$Q_{2n}(\mathbf{A}) = \underbrace{\langle \mathbf{F} \wedge \mathbf{F} \wedge \cdots \wedge \mathbf{F} \rangle}_{\text{n-times}}$$

satisfy these requirements, where we have defined

$$\gamma_{a_1\ldots a_n} \equiv \langle T_{a_1} T_{a_2} \cdots T_{a_n} \rangle \, ,$$

and $\langle \ldots \rangle$ stands for a trace operation in an appropriate representation of the Lie algebra G. The invariants of the form (15) are in one-to-one correspondence with the nth rank invariant tensors $\gamma_{a_1\ldots a_n}$ which could be constructed for a given gauge group. The number of such tensors is rather small in general and is related to the number of Casimir invariants of the group. (In the next two sections we will discuss specific realizations of these brackets, so most of the mystery will be dissipated shortly.) From now on, we will not explicitly write the wedge symbol unless there is an ambiguity, so we will also write Q_{2n} as

$$Q_{2n}(\mathbf{A}) = \langle \mathbf{F}^n \rangle \, . \tag{16}$$

These invariants belong to the family of **characteristic classes** known as the Chern-Weil invariants. These classes themselves define integral invariants related to the topological properties of the maps that can be established between manifold M and the group $\mathbf{G}$.

The equation analogous to (6) now reads

$$dL_{CS}^{2n-1} = \langle \mathbf{F}^n \rangle, \tag{17}$$

and its solution can be written as

$$L_{CS}^{2n-1} = \frac{1}{(n+1)!} \int_0^1 dt \left\langle \mathbf{A}(td\mathbf{A} + t^2\mathbf{A}^2)^{n-1} \right\rangle + \alpha, \tag{18}$$

where α is an arbitrary closed $(2n-1)$-form $(d\alpha = 0)$.

Under a gauge transformation of the form (8), the Chern-Simons form (18) changes as

$$L_{CS}^{2n-1}(\mathbf{A}') = L_{CS}^{2n-1}(\mathbf{A}) + d\beta + (-1)^{n-1} \frac{n!(n-1)!}{(2n-1)!} \left\langle (g^{-1}dg)^{2n-1} \right\rangle, \tag{19}$$

where the $(2n-1)$-form β is a function of $\mathbf{A}$ and depends on g through the combination $g^{-1}dg$. Thus, the action

$$I_{CS}^{2n-1}[\mathbf{A}] = \int_M L_{CS}^{2n-1},$$

describes a gauge theory for the group $\mathbf{G}$, which under a finite gauge transformation changes as the integral of (19). The second term in the RHS of (19) gives rise to a boundary term, and the third is proportional to the winding number. Again, one can see that under an infinitesimal gauge transformation (connected to the identity) of the form

$$\delta\mathbf{A} = -\nabla\lambda, \text{ where } \lambda << 1,$$

the action changes by a surface term, which can be set to zero under the appropriate boundary conditions.

III. GENERAL RELATIVITY AS A CHERN-SIMONS THEORY

So far, the best description of our universe at large scale is general relativity with the Einstein-Hilbert **(EH)** action defined on a four-dimensional spacetime

$$I = \int_{M_4} \sqrt{-g}(R - 2\Lambda)d^4x, \tag{20}$$

where R is the Ricci scalar curvature and Λ is the cosmological constant. More than sixty years of frustrated efforts to quantize this theory can explain the immediate attention drawn by Witten's classical observation that gravity in 2+1 spacetime dimensions is an exactly solvable model [11]. This result means that the quantum theory can be completely and explicitly spelled out, that is, all correlation functions or, alternatively, its entire Hilbert space, can be known. This is remarkable since in just one more spatial dimension, the quantization problem becomes intractable. It could be argued that quantization of 2+1 gravity is no big deal since the theory has no propagating degrees of freedom and therefore its quantum description is like that of a system of point particles. Although this is certainly an important simplification, the key to the proof of solvability is the fact that 2+1 gravity is a **CS** system and hence, it has all the nice features of a gauge theory. Gravity in 3+1 dimensions, on the other hand cannot be construed as a gauge system of the Poincaré or (A-)dS groups, and this is a serious limitation for it quantization. In what follows we will cast 2+1 gravity as a CS system and will see how this can be generalized to higher dimensions.

The three-dimensional EH action analogous to (20) can be cast as a first order theory, in which only first order derivatives occur, by using form language,

$$I[\omega, e] = \frac{1}{2} \int \epsilon_{abcd} \left(R^{ab} - \frac{1}{6}\Lambda e^a e^b \right) e^c e^d. \tag{21}$$

Here R^{ab} is the curvature two-form $R^a_b = d\omega^a_b + \omega^a_c\omega^c_b$ (here the wedge product symbol ($\wedge$) has been suppressed), and the dynamical fields are the vielbein $e^a = e^a_\mu(x)dx^\mu$, and the Lorentz (spin) connection $\omega^{ab}_\mu = \omega^{ab}_\mu(x)dx^\mu$. Note that the action (21) depends on the fields and their *first derivatives only* (this is a consequence of the fact that only exterior derivatives are used throughout and therefore higher order derivatives with respect to one coordinate can never occur). When this action is varied with respect to e^d and ω^{ab}, the following field equations are found

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$$\epsilon_{abcd}\left(R^{ab} - \frac{1}{3}\Lambda e^a e^b\right)e^c = 0 \tag{22}$$

$$\epsilon_{abcd}T^c e^d = 0. \tag{23}$$

The first expression are the usual Einstein's equation, which relates the curvature to the metric, while the second implies vanishing torsion, $T^a := De^a = de^a + \omega^a_{\ b}e^b = 0$, which is usually postulated in general relativity. Solving this second equation for ω^{ab} as a function of e, de and e^{-1} and substituting $\omega^{ab}(e)$ into (22) the standard second order form of the Einstein equations is obtained. The action (21) is the first order formulation of the EH theory and it is the most general local 4-form invariant under Lorentz rotations in the tangent space, constructed out of the vielbein, the spin connection and their exterior derivatives only [13,4]. This suggests that in other dimensions, the same prescription could be applied to construct the action for the gravitational field.

A. Beyond the Einstein-Hilbert action

In order to describe the gravitational field for $d > 4$ one could assume the spacetime geometry as given by the same **EH** action (20) –with or without cosmological constant– integrated now over d dimensions. One of the reasons for the universal appeal of the EH action is that it yields second order field equations, but as we already mentioned, other attractive alternatives exist in higher dimensions of the form (1).

The Lanczos-Lovelock (**LL**) Lagrangian is the most general local d-form invariant under Lorentz rotations of the tangent space, constructed out of the vielbein, the spin connection and their exterior derivatives without using other structures [13]. In particular, the metric, the inverse vielbein or the Hodge-* dual[4] are never used in the construction, ensuring that only first order field equations for e and ω can be produced. If the torsion-free condition is assumed, the equations for the metric become second order.

B. Gravity as an (anti-)de Sitter CS Theory

The coefficients in the **LL** action can be made dimensionless by the redefinition $\alpha_p \to \alpha_p l^{d-2p}$, where l is a parameter with dimensions of length. Then, the **LL** Lagrangian reads

$$L = \sum_{p=0}^{[d/2]} \alpha_p l^{2p-d} \epsilon_{a_1 \ldots a_d} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d}. \tag{24}$$

The embarrassing freedom to choose the α_p's arbitrarily can be drastically cut by the following observation: The field equations for e and ω involve R^{ab}, T^a and e^a. Taking the covariant exterior derivatives of these equations and using the Bianchi identities, new algebraic relations between the curvature and torsion tensors are produced, which would generically introduce nonholonomic restrictions on the degrees of freedom of the theory . This is so for all choices of α_p's, except in two cases, for which no new constraints on the geometry are generated [14]:

- $d = 2n$: The Born-Infeld (**BI**) case,

$$\alpha_p = \kappa \begin{pmatrix} n \\ p \end{pmatrix}, o \leq p \leq n \tag{25}$$

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[4]The Hodge $*$-operation relates a p-form and a $(d-p)$-form through its action on the basis

$$*(dx^{\mu_1} \cdots dx^{\mu_p}) = \frac{1}{(d-p)!}\sqrt{|g|}g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p} \times$$
$$\epsilon_{\nu_1 \cdots \nu_d} dx^{\nu_{p+1}} \cdots dx^{\mu_d}.$$

Note that this expression involves explicitly the inverse of the metric and its inverse.

- $d = 2n - 1$: The AdS Chern-Simons (**AdS-CS**) case,

$$\alpha_p = \frac{\kappa}{d - 2p} \binom{n-1}{p}, 0 \leq p \leq n - 1. \tag{26}$$

The reference to BI in the first case stems from the fact that, in that case, the Lagrangian reads

$$L = \kappa \cdot \epsilon_{a_1 \ldots a_d} \left(R^{a_1 a_2} + \frac{1}{l^2} e^{a_1} e^{a_2} \right) \cdots \left(R^{a_{d-1} a_d} + \frac{1}{l^2} e^{a_{d+1}} e^{a_d} \right),$$

$$= \kappa \cdot pfaff \left[R^{ab} + \frac{1}{l^2} e^a e^b \right], \tag{27}$$

$$= \kappa \cdot \sqrt{\det \left[R^{ab} + \frac{1}{l^2} e^a e^b \right]},$$

which is reminiscent of Lagrangian for the Born-Infeld electrodynamics

The odd-dimensional case is referred to as AdS-CS because that Lagrangian is of the CS family and it can be cast in a form which is manifestly invariant under local AdS transformations. This can be seen as follows. Consider the array of 1-forms

$$W^{AB} = \begin{bmatrix} \omega^{ab} & l^{-1} e^a \\ -l^{-1} e^b & 0 \end{bmatrix}, \tag{28}$$

where the indices $a, b, \ldots = 1, 2, \ldots d$ and $A, B, \ldots = 1, 2, \ldots d + 1$. It is a simple exercise to show that this connection defines a curvature 2-form given by

$$\overline{R}^{AB} = dW^{AB} + W_C^A W^{CB}$$

$$= \begin{bmatrix} R^{ab} + l^{-2} e^a e^b & l^{-1} T^a \\ -l^{-1} T^b & 0 \end{bmatrix}, \tag{29}$$

where the $A, B, \ldots$ indices are raised and lowered using the AdS metric[5],

$$\eta_{AB} = \text{diag}(-1, -1, +1, \ldots, +1). \tag{30}$$

The AdS curvature $\overline{R}^{AB}$ can be used to construct the **Euler form** in $2n$ dimensions as

$$\mathcal{E}_{2n} = \epsilon_{A_1 \ldots A_{2n}} \overline{R}^{A_1 A_2} \cdots \overline{R}^{A_{2n d-1} A_{2n}}, \tag{31}$$

which, by virtue of the Bianchi identity ($D\overline{R}^{ab} = 0$), can be shown to be closed, $d\mathcal{E}_{2n} = 0$. [Here $\overline{R}^{AB}$ denotes the $(d+1 = 2n)$-dimensional curvature, not to be confused with the $(d = 2n-1)$-dimensional one, R^{ab}.] Substituting (29) into (31) one finds the explicit form for $\mathcal{E}_{2n}$ in terms of the $(d = 2n-1)$-dimensional tensors under the $SO(2n-2, 1)$ (Lorentz) group,

$$\mathcal{E}_{2n} = n\epsilon_{a_1 \ldots a_{2n-1}} (R^{a_1 a_2} + l^{-2} e^{a_1} e^{a_2}) \cdots (R^{a_{2n d-3} a_{2n-2}} + l^{-2} e^{a_{2n d-3}} e^{a_{2n-2}}) T^{a_{2n-1}}.$$

Now, the same arguments applied to find the CS Lagrangian from the Pontryagin density can be applied now to the Euler form. Thus, one may ask, what is the $(2n-1)$-form whose exterior derivative gives (31)?. Direct computation yields the answer in the form (24), but with fixed coefficients:

$$\mathcal{E}_{2n} = d \left(n \sum_{p=0}^{[d/2]} \frac{\kappa l^{2p-d}}{d - 2p} \binom{n-1}{p} \epsilon_{a_1 \ldots a_d} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d} \right). \tag{32}$$

$$= dL_{2n-1}^{AdS}. \tag{33}$$

That is, the coefficients α_p given by (26).

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[5]All that it is said in this lecture about AdS can be easily carried over to the de Sitter case. It is necessary to change a few signs at the right places.

C. Gauge invariance

We now show that the Lagrangian L^{AdS}_{2n-1} is really invariant under infinitesimal $SO(d-1,2)$ (AdS) rotations. Under $SO(d-1,1)$ e^a and ω^a_b transform as

$$
\begin{aligned}
\omega^{ab} \to \omega'^{ab} &= d\lambda^{ab} + \omega^{ac}\lambda^b{}_c + \omega^{cb}\lambda^a{}_c \\
&\equiv D_\omega \lambda^{ab} \\
e^a \to e'^a &= \lambda^a{}_b e^b
\end{aligned}
\qquad (34)
$$

then the new curvature $\overline{R}^{AB}$ transforms as a tensor under a larger group. In fact, (34) is a particular case of the $SO(d-1,2)$ transformations of the form

$$
\begin{aligned}
W^{AB} \to W'^{AB} &= d\Lambda^{AB} + W^A_C \Lambda^{CB} + W^B_C \Lambda^{AC} \\
&\equiv D_W \Lambda^{AB}
\end{aligned}
\qquad (35)
$$

where we have defined

$$
\Lambda^{AB} = \begin{bmatrix} \lambda^{ab} & l^{-1}\lambda^a \\ -l^{-1}\lambda^b & 0 \end{bmatrix}.
\qquad (36)
$$

These transformations include, besides the Lorentz transformations (determined by λ^{ab}), also "AdS boosts" defined by λ^a. Setting $\lambda^a = 0$ in (36), (34) is obtained. Therefore, the $(d+1)$-form $\mathcal{E}_{2n}$ is invariant under the full Anti-de Sitter group. This in turn implies that the CS Lagrangian obtained from the Euler form is also invariant under local AdS transformations. It is the magic of the choice (26) that the group of local invariances of the action has grown from $SO(d-1,1)$ to $SO(d-1,2)$.

Finally, the Lagrangian in $d = 2n - 1$ dimensions reads

$$
L_{2n-1} = \kappa \sum_{p=0}^{n-1} \frac{1}{D-2p} \binom{n-1}{p} l^{2p-D} \epsilon_{a_1 \cdots a_{2n-1}} R^{a_1 a_2} \cdots R^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \cdots e^{a_{2n-1}}.
\qquad (37)
$$

Here κ is the gravitational constant similar to Newton's constant in $d = 4$ and can be shown to be quantized in this theory [15]. A remarkable feature of this Lagrangian is that if written in terms of the AdS connection W^{AB}, it has no dimensionful constants, and this makes it a good candidate for a renormalizable field theory. However, there is an obscure point here because the vacuum of the theory seems to be defined by the **AdS-flat** configuration $W^{AB} = 0$ and this means that both ω^{ab} and e^a must vanish, a situation hard to reconcile with a meaningful spacetime interpretation.

D. Field Equations

The field equations obtained varying with respect to W^{AB} are

$$
\epsilon_{A_1 \ldots A_{2n}} \overline{R}^{A_3 A_4} \cdots \overline{R}^{A_{2nd-1}A_{2n}} = 0,
$$

or, equivalently varying with respect to ω^{ab} and e^a

$$
\begin{aligned}
\delta\omega^{ab}: \quad & \epsilon_{aba_3 \ldots a_{2n-1}} (R^{a_3 a_4} + l^{-2} e^{a_3} e^{a_4}) \cdots (R^{a_{2nd-3}a_{2n-2}} + l^{-2} e^{a_{2nd-3}} e^{a_{2n-2}}) T^{a_{2n-1}} = 0 \\
\delta e^a: \quad & \epsilon_{aa_2 \ldots a_{2n-1}} (R^{a_2 a_3} + l^{-2} e^{a_2} e^{a_3}) \cdots (R^{a_{2nd-2}a_{2n-1}} + l^{-2} e^{a_{2nd-2}} e^{a_{2n-1}}) = 0
\end{aligned}
\qquad (38)
$$

Configurations with *vanishing local AdS curvature* $(\overline{R}^{AB} = 0)$ are obvious solutions of these equations, while torsion-free spaces with *flat Lorentz curvature* $(T^a = 0 = R^{ab})$ *are not*. Other less trivial solutions are torsion-free spaces with $(2n-2)$-dimensional submanifolds of constant curvature foliated along one direction, such as black holes or cosmological solutions [16].

E. Gravity in 2+1 Dimensions

As in all CS systems, the 2+1 dimensional case is illuminating and not completely trivial. In this case the **CS** Lagrangian whose exterior derivative is the Euler density in 4 dimensions is the standard EH action in three dimensions,

$$L_{2+1} = \frac{\kappa}{l}\epsilon_{abc}(R^{ab} + \frac{1}{3l^2}e^a e^b)e^c. \tag{39}$$

The corresponding field equations are

$$\epsilon_{abc}(R^{ab} + \frac{1}{l^2}e^a e^b) = 0, \tag{40}$$

$$T^a = 0. \tag{41}$$

The last equation implies that the connection can be written as a function of the vielbein and (40) states that the spacetime must have constant curvature at each point. In view of the fact that (2+1) gravity has no propagating degrees of freedom, constant curvature spacetimes were thought to be rather dull configurations. However, one can be surprised by the fact that solving (40) explicitly for spherically symmetric, static configurations does not necessarily produce a globally AdS spacetime, but an *AdS spacetime with identifications* as well. This is because the field equations only refer to local properties of spacetime and do not restrict the global topology further. Thus, the spacetime manifold can be cut and pasted, identifying points connected by a **finite isometry** along a Killing vector –as when one makes a cylinder out of a plane–, one should still have a solution[6]. In fact, one can produce a black hole in this fashion . What is even more remarkable, is the fact that one can generate a solution by a Lorentz boost that sets the black hole in rotation about its symmetry axis. [17]

A good example of a nontrivial solution of (40) is the 2+1 black hole geometry [18,19],

$$ds^2 = -N(r)^2 f(r)^2 dt^2 + f(r)^{-2}dr^2 + r^2(d\varphi + N^\varphi(r)dt)^2,$$
$$0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi, \quad t_1 \leq t \leq t_2. \tag{42}$$

Solving the Einstein equations yields

$$f^2 = r^2 - M + \frac{J^2}{4r^2} \tag{43}$$

$$N = N(\infty) \tag{44}$$

$$N^\varphi = -\frac{J}{2}\left(\frac{1}{r^2} + N^\varphi(\infty)\right), \tag{45}$$

which defines a black hole of mass M, angular momentum J, provided $J^2 \leq M^2$.

F. Exotic Gravity

The Pontryagin form can be defined for any group, and in particular this is also true of the Lorentz and AdS groups too. The Lorentz-Pontryagin form is defined as

$$P^{Lor} = R^{a_1}_{a_2}R^{a_2}_{a_3}\cdots R^{a_n}_{a_1}, \tag{46}$$

and the AdS-Pontryagin form as

$$P^{AdS} = \overline{R}^{A_1}_{A_2}\overline{R}^{A_2}_{A_3}\cdots \overline{R}^{A_n}_{A_1}. \tag{47}$$

Because these curvature two-forms are antisymmetric in their indices, it is obvious that they can only be defined for even n. This means that they are naturally constructed in $4k$ dimensions. This in turn means that both AdS and Lorentz Chern-Simons theories associated with the Pontryagin family can only exist in $4k - 1$ dimensions, these are the so-called **exotic Lagrangians**, which are independent from the Euler CS forms discussed above in $d = 2n - 1$.

---

[6]Care should be taken, however not to produce closed timelike curves that could generate paradoxical spacetimes scenarios (where one could kill his ancestors, for example). Spaces with closed timelike curves of a *large finite* proper length only could still make sense: if the period of those curves is of the order of the age onf the universe, for instance.

IV. CHERN-SIMONS SUPERGRAVITY

Supersymmetry is the only nontrivial way to extend spacetime symmetries. This result was well known before the era of supersymmetry and it states that in a local field theory whose S-matrix relates in- and out- eigenstates of energy and momentum, all other "internal" quantum numbers such as color, flavor, hypercharge, etc., must be spacetime scalars. In other words, under spacetime transformations these labels do not transform. In mathematical terms this means that the group of invariances of the S-matrix must be of the form $G = \mathcal{S} \otimes \mathcal{I}$, where $\mathcal{S}$ is the group of spatial transformations (e.g., Poincaré, AdS, etc.), and $\mathcal{I}$ is the internal symmetry group.

For some time it was hoped that the nonrenormalizability of GR could be cured by its supersymmetric extension. However, the initial hopes raised by supergravity (**SUGRA**) as a mechanism for taming the ultraviolet divergences of pure gravity eventually vanished with the realization that SUGRAs would be nonrenormalizable as well [20]. Again, one can see that, like GR, SUGRA is not a gauge theory for a group or a supergroup, and that the local (super-) symmetry algebra closes naturally only on shell. The algebra could be made to close off shell by force, at the cost of introducing auxiliary fields –which are not guaranteed to exist for all d and N and [21]–, and still the theory would not have a fiber bundle structure since the base manifold is identified with part of the fiber.

Whether it is the lack of fiber bundle structure the ultimate reason for the nonrenormalizability of gravity remains to be proven. It is certainly true, however, that if GR could be formulated as a gauge theory, the chances for proving its renormalizability would clearly grow.

In three spacetime dimensions, on the other hand, both GR and SUGRA define renormalizable quantum theories. It is strongly suggestive that precisely in 2+1 dimensions both theories can also be formulated as gauge theories on a fiber bundle. It could be thought that the exact solvability miracle is due to the absence of propagating degrees of freedom in three-dimensional gravity, but final the power counting renormalizability argument rests on the fiber bundle structure of the Chern-Simons form of those systems.

There are other known examples of gravitation theories in odd dimensions which are genuine (off-shell) gauge theories for the anti-de Sitter (**AdS**) or Poincaré groups [22–24,15]. These theories, as well as their supersymmetric extensions have propagating degrees of freedom [25] and are CS systems for the corresponding groups as shown in [26].

A. From Rigid Supersymmetry to Supergravity

Rigid SUSY can be understood as an extension of the Poincaré algebra by including supercharges which are the "square roots" of the generators of rigid translations, $\{\bar{Q}, Q\} \sim \Gamma \cdot P$. This idea can be extended to local SUSY substituting the momentum $P_\mu = i\partial_\mu$ by the generators of diffeomorphisms, $\mathcal{H}$, and relating them to the supercharges by $\{\bar{Q}, Q\} \sim \Gamma \cdot \mathcal{H}$. The resulting theory has on-shell local supersymmetry algebra [27].

An alternative approach –which we would like to advocate here– is to construct the supersymmetry *on the tangent space and not on the base manifold*. This point of view is more natural if one recalls that spinors provide a basis of irreducible representations for $SO(N)$, and not for $GL(N)$. Thus, spinors are naturally defined relative to a local frame on the tangent space rather than on the coordinate basis. The basic point is to reproduce the 2+1 "miracle" in higher dimensions. This idea has been successfully applied in five dimensions [23], and for pure gravity [24,15] and to supergravity [28,26]. The SUGRA construction has been carried out for spacetimes whose tangent space has AdS symmetry [28], and for its Poincaré contraction in [26].

B. Assumptions of Standard Supergravity

Three implicit assumptions are usually made in the construction of standard SUGRA:

(i) The fermionic and bosonic fields in the Lagrangian should come in combinations such that their propagating degrees of freedom are equal in number. This is usually achieved by adding to the graviton and the gravitini a number of lower spin fields ($s < 3/2$) [27]. This matching, however, is not necessarily true in AdS space, nor in Minkowski space if a different representation of the Poincaré group (e.g., the adjoint representation) is used [29].

The other two assumptions concern the purely gravitational sector. They are as old as General Relativity itself and are dictated by economy: **(ii)** gravitons are described by the Hilbert action (plus a possible cosmological constant), and, **(iii)** the spin connection and the vielbein are not independent fields but are related through the torsion equation. The fact that the supergravity generators do not form a closed off-shell algebra can be traced back to these assumptions..

The procedure behind **(i)** is tightly linked to the idea that the fields should be in a *vector* representation of the Poincaré group and that the kinetic terms and couplings are such that the counting of degrees of freedom works like

in a minimally coupled gauge theory. This assumption comes from the interpretation of supersymmetric states as represented by the in- and out- plane waves in an asymptotically free, weakly interacting theory in a minkowskian background. These conditions are not necessarily met by a CS theory in an asymptotically AdS background. Apart from the difference in background, which requires a careful treatment of the unitary irreducible representations of the asymptotic symmetries [30], the counting of degrees of freedom in CS theories is completely different from the counting for the same connection 1-forms in a YM theory.

Thus, the only natural extension for a gauge theory of the AdS spacetime is supergravity, constructed enlarging the group to some supergroups that contain AdS but are otherwise semisimple. The construction of these theories can be found in [31]. The crucial observation is that the Dirac matrices provide a natural representation of the AdS algebra in any dimension, thus, the connection W^{AB} can be written in this representation as $W = e^a J_a + \frac{1}{2}\omega^{ab} J_{ab}$, where

$$J_a = \begin{bmatrix} \frac{1}{2}(\Gamma_a)^\alpha_\beta & 0 \\ 0 & 0 \end{bmatrix}, \tag{48}$$

$$J_{ab} = \begin{bmatrix} \frac{1}{2}(\Gamma_{ab})^\alpha_\beta & 0 \\ 0 & 0 \end{bmatrix}. \tag{49}$$

This spinorial representation for the connection naturally leads to a representation for a superalgebra whose generators have entries in the remaining blocks of similar matrices. The non zero blocks in (48) and (49) are $m \times m$ where $m = 2^{[d/2]}$ is the number of components of a spinor in d dimensions. The algebra is completed by the (pseudo-) Majorana generators of supersymmetry,

$$Q^k_\gamma = \begin{bmatrix} 0 & \delta^\alpha_\gamma \delta^k_j \\ -C_{\gamma\beta} u^{ki} & 0 \end{bmatrix}, \tag{50}$$

which are in a vector representation (described by the index k) of some internal symmetry group with generators

$$M^{kl} = \begin{bmatrix} 0 & 0 \\ 0 & (m^{kl})^i_j \end{bmatrix}. \tag{51}$$

Following [31], one arrives at all the possible superalgebras in dimensions d, except for $d = 5 \mod \dot{4}$. In those cases the representations are necessarily complex. Then the pseudo-Majorana condition has to be relaxed and the generators of supersymmetry take the form

$$\bar{Q}^l_\gamma = \begin{bmatrix} 0 & \delta^\alpha_\gamma \delta^l_j \\ 0 & 0 \end{bmatrix}, \tag{52}$$

$$Q_{\rho k} = \begin{bmatrix} 0 & 0 \\ -G_{\rho\beta} \delta^i_k & 0 \end{bmatrix}, \tag{53}$$

instead of (50). With this, one can write the algebra for any d. The only nontrivial condition for the closure of the superalgebra comes from the anticommutator of two supersymmetry generators. In some cases this anticommutator includes all generators of the Pauli algebra, $\Gamma_{(k)} = \frac{1}{k!}\delta^{a_1 \cdots a_k}_{b_1 \cdots b_k}\Gamma^{b_1} \cdots \Gamma^{b_k}$, $0 \le k \le d$ (for $d = 5 \mod 4$), sometimes it contains only the symmetric subalgebra, $(C\Gamma_{(k)})^T = C\Gamma_{(k)}$ (for $d = 2, 6, 7, 8 \mod 8$), and sometimes it includes only the antisymmetric ones, $(C\Gamma_{(k)})^T = -C\Gamma_{(k)}$ (for $d = 2, 3, 4, 6 \mod 8$). The fact that $d = 2$ and 6 appear in both families is due to the fact that in those cases there are two inequivalent choices of charge conjugation matrix (C) with $C^T = \pm C$.

We now consider the supersymmetric extensions of the locally AdS theories defined above. In particular, one can write the "exotic" Lagrangian,

$$d\widehat{L}_{4k-1} = \frac{-1}{2^{4k}} Tr[(R^{AB}\Gamma_{AB})^{2k}]. \tag{54}$$

which is a particular form of the Pontryagin form (47). Other possibilities of the form $Tr[\mathbf{F}^{n-p}]Tr[\mathbf{F}^p]$, are not necessary to reproduce the minimal supersymmetric extensions of AdS containing the Hilbert action. In the supergravity theories discussed below, the gravitational sector is given by

$$dL_{2n-1} = STr[F$$

$$dL_{2n-1}^{Grav} = \frac{-1}{2^{4k}} Tr[(R^{AB}\Gamma_{AB})^{2k}] \tag{55}$$

$$= \pm \frac{1}{2^n} L_{G\ 2n-1}^{AdS} - \frac{1}{2} L_{T\ 2n-1}^{AdS}. \tag{56}$$

which is a particular form of (47) where the trace over spinor indices in this representation. Other possibilities of the form $\langle \mathbf{F}^{n-p}\rangle\langle\mathbf{F}^p\rangle$, are not necessary to reproduce the minimal supersymmetric extensions of AdS containing the Hilbert action. The $\pm$ signs correspond to the two choices of inequivalent representations of Γ's, which in turn reflect the two chiral representations in $d+1$. As in the three-dimensional case, the supersymmetric extensions of L_{2n-1} or any of the exotic Lagrangians such as $\widehat{L}_{2n-1}$, require using both chiralities, thus doubling the algebras. Here we choose the $+$ sign, which gives the minimal superextension.

Under a gauge transformation, $\mathbf{A}$ transforms by $\delta\mathbf{A} = \nabla\lambda$, where ∇ is the covariant derivative for the same connection $\mathbf{A}$. In particular, under a supersymmetry transformation, $\lambda = \bar{\epsilon}^i Q_i - \bar{Q}^i\epsilon_i$, and

$$\delta_\epsilon\mathbf{A} = \begin{bmatrix} \epsilon^k\bar{\psi}_k - \psi^k\bar{\epsilon}_k & D\epsilon_j \\ -D\bar{\epsilon}^i & \bar{\epsilon}^i\psi_j - \bar{\psi}^i\epsilon_j \end{bmatrix}, \tag{57}$$

where D is the covariant derivative on the bosonic connection, $D\epsilon_j = (d + \frac{1}{2}[e^a\Gamma_a + \frac{1}{2}\omega^{ab}\Gamma_{ab} + \frac{1}{r!}b^{[r]}\Gamma_{[r]}])\epsilon_j - a_j^i\epsilon_i$.

Two interesting cases can be mentioned here:

C. d=5 SUGRA

In this case the supergroup is $U(2,2|N)$. The associated connection can be written as,

$$\mathbf{A} = e^a\mathbf{J}_a + \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + a^\Lambda\mathbf{T}_\Lambda + (\bar{\psi}^r\mathbf{Q}_r - \bar{\mathbf{Q}}^r\psi_r) + b\mathbf{Z},$$

where the generators $\mathbf{J}_a$, $\mathbf{J}_{ab}$, form an AdS algebra ($so(4,2)$), $\mathbf{T}_\Lambda$ ($\Lambda = 1, \cdots N^2 - 1$) are the generators of $su(N)$, $\mathbf{Z}$ generates a $U(1)$ subgroup and $\mathbf{Q}, \bar{\mathbf{Q}}$ are the supersymmetry generators, which transform in a vector representation of $SU(N)$. The Chern-Simons Lagrangian for this gauge algebra is defined by the relation $dL = iSTr[\mathbf{F}^3]$, where $\mathbf{F} = \mathbf{dA} + \mathbf{A}^2$ is the (antihermitean) curvature. Using this definition, one obtains the Lagrangian originally discussed by Chamseddine in [23],

$$L = L_G(\omega^{ab}, e^a) + L_{su(N)}(a_s^r) + L_{u(1)}(\omega^{ab}, e^a, b) + L_F(\omega^{ab}, e^a, a_s^r, b, \psi_r), \tag{58}$$

with

$$\begin{aligned} L_G &= \tfrac{1}{8}\epsilon_{abcde}\left[R^{ab}R^{cd}e^e/l + \tfrac{2}{3}R^{ab}e^ce^de^e/l^3 + \tfrac{1}{5}e^ae^be^ce^de^e/l^5\right] \\ L_{su(N)} &= -Tr\left[a(da)^2 + \tfrac{3}{2}a^3da + \tfrac{3}{5}a^5\right] \\ L_{u(1)} &= \left(\tfrac{1}{4^2} - \tfrac{1}{N^2}\right)b(db)^2 + \tfrac{3}{4l^2}\left[T^aT_a - R^{ab}e_ae_b - l^2R^{ab}R_{ab}/2\right]b\,, \\ &\quad + \tfrac{3}{N}f_s^r f_r^s b \\ L_F &= \tfrac{3}{2i}\left[\bar{\psi}^r\mathcal{R}\nabla\psi_r + \bar{\psi}^s\mathcal{F}_s^r\nabla\psi_r\right] + c.c. \end{aligned} \tag{59}$$

where $a_s^r \equiv a^\Lambda(\tau_\Lambda)_s^r$ is the $su(2,2)$ connection, f_s^r is its curvature, and the bosonic blocks of the supercurvature: $\mathcal{R} = \frac{1}{2}T^a\Gamma_a + \frac{1}{4}(R^{ab} + e^ae^b)\Gamma_{ab} + \frac{i}{4}dbI - \frac{1}{2}\psi_s\bar{\psi}^s$, $\mathcal{F}_s^r = f_s^r + \frac{i}{N}db\delta_s^r - \frac{1}{2}\bar{\psi}^r\psi_s$. The cosmological constant is $-l^{-2}$, and the AdS covariant derivative ∇ acting on ψ_r is

$$\nabla\psi_r = D\psi_r + \frac{1}{2l}e^a\Gamma_a\psi_r - a_r^s\psi_s + i\left(\frac{1}{4} - \frac{1}{N}\right)b\psi_r. \tag{60}$$

where D is the covariant derivative in the Lorentz connection.

The above relation implies that the fermions carry a $u(1)$ "electric" charge given by $e = \left(\frac{1}{4} - \frac{1}{N}\right)$. The purely gravitational part, L_G is equal to the standard Einstein-Hilbert action with cosmological constant, plus the dimensionally continued Euler density[7].

---

[7]The first term in L_G is the dimensional continuation of the Euler (or Gauss-Bonnet) density from two and four dimensions, exactly as the three-dimensional Einstein-Hilbert Lagrangian is the continuation of the the two dimensional Euler density. This is the leading term in the limit of vanishing cosmological constant ($l \to \infty$), whose local supersymmetric extension yields a nontrivial extension of the Poincaré group [26].

The action is by construction invariant –up to a surface term– under the local (gauge generated) supersymmetry transformations $\delta_\lambda A = -(d\lambda + [A, \lambda])$ with $\lambda = \bar{\epsilon}^r \mathbf{Q}_r - \bar{\mathbf{Q}}^r \epsilon_r$, or

$$
\begin{aligned}
\delta e^a &= \tfrac{1}{2} \left(\bar{\epsilon}^r \Gamma^a \psi_r - \bar{\psi}^r \Gamma^a \epsilon_r \right) \\
\delta \omega^{ab} &= -\tfrac{1}{4} \left(\bar{\epsilon}^r \Gamma^{ab} \psi_r - \bar{\psi}^r \Gamma^{ab} \epsilon_r \right) \\
\delta a^r_{\ s} &= -i \left(\bar{\epsilon}^r \psi_s - \bar{\psi}^r \epsilon_s \right) \\
\delta \psi_r &= -\nabla \epsilon_r \\
\delta \bar{\psi}^r &= -\nabla \bar{\epsilon}^r \\
\delta b &= -i \left(\bar{\epsilon}^r \psi_r - \bar{\psi}^r \epsilon_r \right).
\end{aligned}
$$

As can be seen from (59) and (60), for $N = 4$ the b field looses its kinetic term and decouples from the fermions (the gravitino becomes uncharged with respect to the $U(1)$ field). The only remnant of the interaction with the b field is a dilaton-like coupling with the Pontryagin four forms for the AdS and $SU(N)$ groups (in the bosonic sector). As it is also shown in the Appendix A, the case $N = 4$ is also special at the level of the algebra, which becomes a superalgebra with a $u(1)$ central extension.

In the bosonic sector, for $N = 4$, the field equation obtained from the variation with respect to b states that the Pontryagin four form of AdS and $SU(N)$ groups are proportional. Consequently, if the curvatures approach zero sufficiently fast at spatial infinity, there is a conserved topological current which states that, for the spatial section, the second Chern characters of AdS and $SU(4)$ are proportional. Consequently, if the spatial section has no boundary, the corresponding Chern numbers are related. Using the fact that $\Pi_4(SU(4)) = 0$, the above implies that the Hirzebruch signature plus the Nieh-Yan number of the spatial section cannot change in time.

D. d=11 SUGRA

In this case, the smallest AdS superalgebra is $osp(32|1)$ and the connection is $\mathbf{A} = \frac{1}{2}\omega^{ab}J_{ab} + e^a J_a + \frac{1}{5!}b^{abcde}J_{abcde} + \bar{Q}\psi$, where b is a totally antisymmetric fifth-rank Lorentz tensor one-form. Now, in terms of the elementary bosonic and fermionic fields, the CS form in (15) reads

$$
L_{11}^{osp(32|1)}(\mathbf{A}) = L_{11}^{sp(32)}(\Omega) + L_F(\Omega, \psi), \tag{61}
$$

where $\Omega \equiv \frac{1}{2}(e^a \Gamma_a + \frac{1}{2}\omega^{ab}\Gamma_{ab} + \frac{1}{5!}b^{abcde}\Gamma_{abcde})$ is an $sp(32)$ connection. The bosonic part of (61) can be written as

$$
L_{11}^{sp(32)}(\Omega) = 2^{-6} L_{G\,11}^{AdS}(\omega, e) - \frac{1}{2} L_{T\,11}^{AdS}(\omega, e) + L_{11}^b(b, \omega, e).
$$

The fermionic Lagrangian is

$$
\begin{aligned}
L_F = {}&6(\bar{\psi}R^4 D\psi) - 3\left[(D\bar{\psi}D\psi) + (\bar{\psi}R\psi) \right](\bar{\psi}R^2 D\psi) \\
&-3\left[(\bar{\psi}R^3\psi) + (D\bar{\psi}R^2 D\psi) \right](\bar{\psi}D\psi) + \\
&2\left[(D\bar{\psi}D\psi)^2 + (\bar{\psi}R\psi)^2 + (\bar{\psi}R\psi)(D\bar{\psi}D\psi) \right](\bar{\psi}D\psi),
\end{aligned}
$$

where $R = d\Omega + \Omega^2$ is the $sp(32)$ curvature. The supersymmetry transformations (57) read

$$
\delta e^a = \tfrac{1}{8}\bar{\epsilon}\Gamma^a\psi \qquad\qquad \delta\omega^{ab} = -\tfrac{1}{8}\bar{\epsilon}\Gamma^{ab}\psi
$$

$$
\delta\psi = D\epsilon \qquad\qquad \delta b^{abcde} = \tfrac{1}{8}\bar{\epsilon}\Gamma^{abcde}\psi.
$$

Standard eleven-dimensional supergravity [32] is an N=1 supersymmetric extension of Einstein-Hilbert gravity that cannot accommodate a cosmological constant [33,34]. An $N > 1$ extension of this theory is not known. In our case, the cosmological constant is necessarily nonzero by construction and the extension simply requires including an internal $so(N)$ gauge field coupled to the fermions, and the resulting Lagrangian is an $osp(32|N)$ CS form [35].

E. Summary

The supergravities presented here have two distinctive features: The fundamental field is always the connection $\mathbf{A}$ and, in their simplest form, they are pure CS systems (matter couplings are discussed below). As a result, these

theories possess a larger gravitational sector, including propagating spin connection. Contrary to what one could expect, the geometrical interpretation is quite clear, the field structure is simple and, in contrast with the standard cases, the supersymmetry transformations close off shell without auxiliary fields.

Torsion. It can be observed that the torsion Lagrangians (L_T)are odd while the torsion-free terms (L_G) are even under spacetime reflections. The minimal supersymmetric extension of the AdS group in $4k-1$ dimensions requires using chiral spinors of $SO(4k)$ [30]. This in turn implies that the gravitational action has no definite parity, but requires the combination of L_T and L_G as described above. In $D = 4k+1$ this issue doesn't arise due to the vanishing of the torsion invariants, allowing constructing a supergravity theory based on L_G only, as in [23]. If one tries to exclude torsion terms in $4k-1$ dimensions, one is forced to allow both chiralities for $SO(4k)$ duplicating the field content, and the resulting theory has two copies of the same system [36].

Field content and extensions with N>1. The field content compares with that of the standard supergravities in $D = 5, 7, 11$ as follows:

D	Standard supergravity	CS supergravity
5	e_μ^a ψ_μ^α $\psi_{\alpha\mu}$	e_μ^a ω_μ^{ab} ψ_μ^α $\psi_{\alpha\mu}$ b
7	e_μ^a $A_{[3]}$ $\psi_\mu^{\alpha i}$ $a_{\mu j}^i$ λ^α ϕ	e_μ^a ω_μ^{ab} $\psi_\mu^{\alpha i}$ $a_{\mu j}^i$
11	e_μ^a $A_{[3]}$ ψ_μ^α	e_μ^a ω_μ^{ab} ψ_μ^α b_μ^{abcde}

Standard supergravity in five dimensions is dramatically different from the theory presented here, which was also discussed by Chamseddine in [23].

Standard seven-dimensional supergravity is an $N = 2$ theory (its maximal extension is N=4), whose gravitational sector is given by Einstein-Hilbert gravity with cosmological constant and with a background invariant under $OSp(2|8)$ [37,38]. Standard eleven-dimensional supergravity [32] is an N=1 supersymmetric extension of Einstein-Hilbert gravity that cannot accommodate a cosmological constant [33,34]. An $N > 1$ extension of this theory is not known.

In the case presented here, the extensions to larger N are straightforward in any dimension. In $D = 7$, the index i is allowed to run from 2 to $2s$, and the Lagrangian is a CS form for $osp(2s|8)$. In $D = 11$, one must include an internal $so(N)$ field and the Lagrangian is an $osp(32|N)$ CS form [28]. The cosmological constant is necessarily nonzero in all cases.

Spectrum. The stability and positivity of the energy for the solutions of these theories is a highly nontrivial problem. As shown in Ref. [25], the number of degrees of freedom of bosonic CS systems for $D \geq 5$ is not constant throughout phase space and different regions can have radically different dynamical content. However, in a region where the rank of the symplectic form is maximal the theory behaves as a normal gauge system, and this condition is stable under perturbations. As it is shown in [39], there exists a nontrivial extension of the AdS superalgebra with one abelian generator for which anti-de Sitter space without matter fields is a background of maximal rank, and the gauge superalgebra is realized in the Dirac brackets. For example, for $D = 11$ and $N = 32$, the only nonvanishing anticommutator reads

$$\{Q_\alpha^i, \bar{Q}_\beta^j\} = \frac{1}{8}\delta^{ij}\left[C\Gamma^a J_a + C\Gamma^{ab}J_{ab} + C\Gamma^{abcde}Z_{abcde}\right]_{\alpha\beta}$$
$$-M^{ij}C_{\alpha\beta},$$

where M^{ij} are the generators of $SO(32)$ internal group. On this background the $D = 11$ theory has 2^{12} fermionic and $2^{12} - 1$ bosonic degrees of freedom. The (super)charges obey the same algebra with a central extension. This fact ensures a lower bound for the mass as a function of the other bosonic charges [**?**].

Classical solutions. The field equations for these theories in terms of the Lorentz components (ω, e, b, a, ψ) are spread-out expressions for $<\mathbf{F}^{n-1}G_{(a)}>= 0$, where $G_{(a)}$ are the generators of the superalgebra. It is rather easy to verify that in all these theories the anti-de Sitter space is a classical solution , and that for $\psi = b = a = 0$ there exist spherically symmetric, asymptotically AdS standard [16], as well as topological [40] black holes. In the extreme case these black holes can be shown to be BPS states.

Matter couplings. It is possible to introduce a minimal couplings to matter of the form $\mathbf{A}\cdot\mathbf{J}$. For $D = 11$, the matter content is that of a theory with (super-) 0, 2, and 5–branes, whose respective worldhistories couple to the spin connection and the b fields.

Standard SUGRA. Some sector of these theories might be related to the standard supergravities if one identifies the totally antisymmetric part of the contorsion tensor in a coordinate basis, $k_{\mu\nu\lambda}$, with the abelian 3-form, $A_{[3]}$. In 11 dimensions one could also identify the antisymmetric part of b with an abelian 6-form $A_{[6]}$, whose exterior derivative, $dA_{[6]}$, is the dual of $F_{[4]} = dA_{[3]}$. Hence, in $D = 11$ the CS theory possibly contains the standard supergravity as well as some kind of dual version of it.

V. DYNAMICAL CONTENTS OF CHERN SIMONS THEORIES

The physical meaning of a theory is defined by the dynamics it displays both at the classical as well as at the quantum levels. In order to understand this question one should be able to separate the physical degrees of freedom from those which are redundant. In particular, it should be possible –at least in principle– to separate the propagating modes from the gauge degrees of freedom, and from those which do not evolve independently at all (second class constraints). The standard way to proceed is Dirac's constrained Hamiltonian analysis and has been studied in **CS** systems in [25]. Here we summarize this analysis but refer the reader to the original papers for the details.

A. The BGH construction

From the dynamical point of view, a CS system can be described by a Lagrangian of the form [8]

$$L_{2n+1} = l_a^i(A_j^b)\dot{A}_i^a - A_o^a K_a, \tag{62}$$

where the $(2n+1)$-dimensional spacetime has been split into space and time, and

$$K_a = -\frac{1}{2^n n}\gamma_{aa_1....a_n}\epsilon^{i_1...i_{2n}}F_{i_1 i_2}^{a_1}\cdots F_{i_{2n-1}i_{2n}}^{a_n}.$$

The field equations are

$$\Omega_{ab}^{ij}(\dot{A}_j^b - D_j A_0^b) = 0, \tag{63}$$
$$K_a = 0, \tag{64}$$

where

$$\Omega_{ab}^{ij} = \frac{\delta l_b^j}{\delta A_i^a} - \frac{\delta l_a^i}{\delta A_j^b} \tag{65}$$

$$= -\frac{1}{2^{n-1}}\gamma_{aba_2....a_n}\epsilon^{iji_3...i_{2n}}F_{i_3 i_4}^{a_2}\cdots F_{i_{2n-1}i_{2n}}^{a_n}$$

is the symplectic matrix. The passage to the Hamiltonian has the problem that the velocities appear linearly in the Lagrangian, ant therefore there is a number of primary constraints

$$\phi_a^i \equiv p_a^i - l_a^i \approx 0. \tag{66}$$

Besides these there are the secondary constraints $K_a \approx 0$, which can be combined with the ϕ's into the expressions

$$G_a \equiv -K_a + D_i\phi_a^i, \tag{67}$$

which together with the ϕ's form a closed algebra,

$$\begin{aligned}\{\phi_a^i, \phi_b^j\} &= \Omega_{ab}^{ij} \\ \{\phi_a^i, G_b\} &= C_{ab}^c\phi_c^i \\ \{G_a, G_b\} &= C_{ab}^c G_c\end{aligned},$$

where C_{ab}^c are the structure constants of the gauge algebra of the theory. Clearly the G's form a first class algebra which reflects the gauge invariance of the theory, while some of the ϕ's are second class and some are first class, depending on the rank of the symplectic form. Here we face the first serious difficulty with CS theory: the matrix Ω_{ab}^{ij} depends on the field configurations, and therefore its rank cannot be thought of as a constant, but it can change from one region of phase space to another. This issue has been analyzed in the context of some simplified mechanical models and the conclusion is that the degeneracy of the system occurs at surfaces of lower dimensionality in phase space, which can be viewed as end sets of (unstable) initial points or sets of (stable) end points for the evolution. Unless the system is chaotic, it can be expected that generic configurations where the rank of Ω_{ab}^{ij} is maximal should fill most of phase space [12].

There is a second more tractable problem and that is how to separate the first and second class constraints among the ϕ's. In Ref. [25] the following results are shown:

- The maximal rank of Ω_{ab}^{ij} is $2nN - 2n$, where N is the number of generators in the gauge Lie algebra.

- In consequence, there are $2n$ first class constraints among the ϕ's which correspond to the generators of spatial diffeomorphisms $(\mathcal{H}_i)$.

- The generator of timelike reparametrizations $\mathcal{H}_\perp$ is not an independent first class constraint.

Putting all these facts together, one concludes that in a generic configuration, the number of degrees of freedom of the theory is

$$g = (\text{n}° \text{ of coordinates}) - (\text{n}° \text{ of } 1^{st}\text{class constraints}) - \frac{1}{2}(\text{n}° \text{ of second class constraints})$$

$$= 2nN - (N + 2n) - \frac{1}{2}(2nN - 2n) = nN - N - n. \tag{68}$$

This result is somewhat perplexing. For the gravity theory as a CS system for the AdS group, this result gives, in $d = 2n + 1$ dimensions has

$$N = \frac{(2n+1)(2n+2)}{2} = (2n+1)(n+1),$$

and therefore,

$$g_{AdS}^{CS} = n(2n+1)(n+1) - (2n+1)(n+1) - n$$
$$= 2n^3 + n^2 - 3n - 1. \tag{69}$$

This number of propagating degrees of freedom is much larger than the number found in a purely metric theory of gravity in dimension d.

$$g_d^{metric} = \frac{d(d-3)}{2} = \frac{(2n+1)(2n-2)}{2}$$
$$= 2n^2 - n - 1. \tag{70}$$

Thus, for $d \geq 3$ the number of degrees of freedom of the CS theory is much larger than that of the standard metric theory. In particular, for $d = 5$, (69) is 13, while (70) equals 5. Obviously, the extra degrees of freedom must correspond to propagating modes contained in the torsion, which here are independent from the metric degrees of freedom. The precise identification of the propagating degrees freedom, however requires separating the first and second class constraints, inverting the Dirac matrix and eliminating the second class constraints consistently. This objective seems unattainable in general for an arbitrary gauge group.

As it is also shown in [25], an important simplification occurs when the group has an invariant abelian factor. In that case the symplectic matrix Ω_{ab}^{ij} takes a partially block-diagonal form where the kernel has the maximal size allowed by a generic configuration. For example, in five dimensions, those authors show that for a group $G = G_0 \otimes U(1)$, one has a generic configuration if the curvature for the G_0 connection vanish, but the $U(1)$ curvature, $f = db$, is a nondegenerate 2-form,

.

$$\Omega_{ab}^{ij}\Big|_{F^a=0} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\gamma_{ab}\epsilon^{ijkl}f_{kl} \end{pmatrix}.$$

Then, the lower block of Ω_{ab}^{ij} is the Dirac matrix and the constrained Hamiltonian analysis can proceed in the standard way.

It is a nice surprise in the cases of CS supergravities discussed above that it seems that for certain unique choices extended supesymmetries the algebras develop an abelian subalgebra and make the separation of first and second class constraints possible. It is remarkable that in some cases (e.g., for $d = 5$, $N = 4$) the algebra **is not** a direct sum but an algebra with an abelian central extension. In other cases (e.g., for $d = 11$, $N = 32$), the algebra is a direct sum, but the abelian subgroup is not put in by hand but it is a subset of the generators that for that particular extended supersymmetry spontaneously decouples from the rest of the algebra.

B. Final Comments

Chern-Simons theories contain a wealth of other interesting features, starting with its relation to geometry and field theories and knot invariants. The higher-dimensional CS systems remain somewhat mysterious especially because of the difficulties to treat them as quantum theories. However, they have many ingredients that make them likely models to be quantized: They carry no dimensionful couplings; the only parameters they can have must be quantized; in gravity these are the only theories of the Lovelock family that gives rise to black holes with positive specific heat [41] and hence, capable to reach thermal equilibrium with an external heat bath.

Efforts to quantize CS systems seem promising at least in the cases in which the space admits a complex structure so that the symplectic form can be cast as a Kähler form [42]

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